Numerical Methods
For
Image Restoration

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Outline

1. Image Restoration as an inverse problem
2. Image degradation models: noises, blurs
3. Ill-conditioning -> Regularization
4. Variational - MAP approaches
5. Blind Deblurring and Inpainting
6. Motion Deblurring
7. Deblurring of particular images
Image Restoration

AIM: Restore Degraded Images
AIM: Restore Degraded Images
Restoration Problem

Degradation model:

\[ u_0 = k \ast u + n \]

**Goal:** Given \( u_0 \), recover \( u \)  

Inverse Problem
Problems

Model (rule)

Input (cause) \xrightarrow{} X

\Phi(x)

\mathbf{p}(\cdot | x)

Output (effect) \xrightarrow{} Y
Problems

Direct Problem (Degradation: blur + noise)

Input (cause) X

Model (rule)

Output (effect) y

\[ \Phi(x) \]

\[ p(\cdot | x) \]
Problems

Identification Problem (Degradation model estimation)

Model (rule)

\[ \Phi(x) \]
\[ p(\cdot | x) \]

Input (cause) \( x \) \rightarrow \text{Model (rule)} \rightarrow \text{Output (effect)} \( y \)

Identification Problem (Degradation model estimation)
Inverse Problem (Restoration: deblurring + denoising)
Inverse + Identification Problem (Degradation model estimation + Restoration: blind deblurring + denoising)
Inverse problems are everywhere

Reconstruct the density distribution within the human body by means of Xray projections

Inferring seismic properties of the Earth’s interior from surface observations

Invert the observed data in order to derive the internal properties of the Sun

Medical tomography
1970s

Seismic tomography
1980s

Helioseismology
1990s

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Image Restoration
Degradation model

Continuous degradation model:

\[ u_0(x) = \int_{\Omega} k(x, y)u(y)dy + n(x) \quad x \in \Omega \in \mathbb{R}^2 \]

- Perturbed observed image
- Blur and noise-free image
- Data noise

Point Spread Function:
- Linear space-variant blur
Degradation model

Continuous degradation model:

\[ u_0(x) = \int_{\Omega} k(x - y)u(y)dy + n(x) \quad x \in \Omega \subseteq \mathbb{R}^2 \]

Perturbed observed image  Blur and noise-free image  Data noise

Point Spread Function:
Linear space-invariant blur

Integral equation can be expressed as

\[ u_0 = k * u + n \]

convolution

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Noise

Original  salt and pepper  speckle  Gauss-add  Gauss-molt

independent identically distributed (i.i.d.)
Blur

PSFs with sharp edges:

Motion Blur

Out of Focus Blur

PSFs with smooth transitions

Gaussian Blur

Scatter Blur
Two causes for motion blur

- Hand shaking: Blur is the same
- Object motion: Blur is different
Degradation model

Continuous degradation model:

\[ u_0(x) = \int_\Omega \int k(x-y)u(y)dy + n(x) \quad x \in \Omega \subseteq R^2 \]

Discretization yields:

\[ u_0 = k \ast u + n \]

with matrix A block Toeplitz with Toeplitz blocks

severely ill-conditioned \rightarrow \text{regularization}

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Degradation model

Discrete degradation model:

\[ u_0 = Au \]

with matrix \( A \) block Toeplitz with Toeplitz blocks

severely ill-conditioned \( \rightarrow \) regularization

Deblurring:
- Invert low-pass filtering
- Backward diffusion process
- Decrease entropy

- Basic method: TSVD (Truncated Singular Value Decomposition)
- Frequency domain methods: Wiener, Richardson-Lucy, wavelets
Boundary conditions

Zero Dirichlet

Periodic

Reflexive (Neumann)

Antireflexive:
S. Serra-Capizzano, M. Donatelli, et al.

Synthetic Boundary Conditions: Y. Wai, J. Nagy
Ill-conditioning effect

Solution $Au=b$: add 0.1% noise to rhs

$$b = \hat{b} + e$$

$$u = A^{-1}(\hat{b} + e) = A^{-1}\hat{b} + A^{-1}e = \hat{u} + A^{-1}e$$

$u = A^{-1}b$
Variational approaches

Minimize the energy functional:

\[ \hat{u}_\lambda = \arg \min_E (u; k) = \int_\Omega \frac{1}{2} (k * u - u_0)^2 + \lambda \phi(\nabla u) \, d\Omega \]

**Data term**: enforces the match between the sought image and the observed image via the blur model

**Smoothness term**: brings in regularity assumptions about the unknown image

\[
\begin{align*}
\Phi(s) &= s^2 \\
\Phi(s) &= \sqrt{s^2 + \varepsilon^2} \\
\Phi(s) &= \rho^2 \ln(1 + s^2 / \rho^2)
\end{align*}
\]

Tikhonov

TV

Perona – Malik

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Variational approaches as MAP estimation

Maximize the posterior probability:
\[ \hat{u} = \arg \max \ p(u | u_0; k) \]

Minimize the Logarithmic Energy:
\[ \hat{u} = \arg \min E(u | u_0; k) = \arg \min ( -\log p(u | u_0; k) ) \]

But (from Bayes rule):
\[ p(u | u_0; k) \propto p(u_0 | u; k) p(u; k) = p(u_0 | u; k) p(u) \]

Hence:
\[ \hat{u} = \arg \min ( E(u | u_0; k) = E(u_0 | u; k) + E(u) ) \]

i.i.d Gaussian noise + exponential prior for image gradients:
\[ \hat{u}_\lambda = \arg \min \left( E_\lambda(u; k) = \int_{\Omega} (k * u - u_0)^2 + \lambda \phi(|\nabla u|) \ d\Omega \right) \]
General model for variational deblurring

\[ \hat{u}_\lambda = \text{arg min} \left\{ E_\lambda (u; k) = \int_{\Omega} \frac{1}{2} (k * u - u_0)^2 + \lambda \phi(\nabla u) \, d\Omega \right\} \]

\[ = \text{arg min} \left\{ \int_{\Omega} G(x, y, u, \nabla u) \, d\Omega \right\} \quad \text{where} \quad \nabla u = (u_x, u_y) \]

Euler-Lagrange equations (can be linear or non-linear):

\[ \frac{dG}{du} = 0 \text{ in } \Omega \]

\[ \frac{\partial}{\partial x} G_{u_x} + \frac{\partial}{\partial y} G_{u_y} - G_u = 0 \quad (x,y) \in \Omega \]

\[ , \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \]

\[ \nabla_u E_\lambda (u; k) = k^t \ast (k \ast u - u_0) - \lambda D(u) = 0 \]

\[ D(u) = \nabla \cdot (\phi'(\nabla u) \frac{\nabla u}{|\nabla u|}) \]
General model for variational deblurring

$$\nabla_u E_\lambda(u; k) = k^t \ast (k \ast u - u_0) - \lambda D(u) = 0 \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega$$

$$D(u) = \nabla \cdot \left( \phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right)$$

$$\Phi(u) = |\nabla u|^2 \quad \Rightarrow \quad D(u) = \Delta u \quad \text{Tikhonov (linear)}$$

$$\Phi(u) = |\nabla u| \quad \Rightarrow \quad D(u) = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) \quad \text{TV (non-linear)}$$

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Second order non-linear elliptic PDEs
**General model for variational deblurring**

\[ u(t = 0) = u_0 \text{ in } \Omega \]

\[ u_t = -\nabla u E_\lambda(u; k) = -k^t * (k * u - u_0) + \lambda D(u) = 0 \text{ in } \Omega \]

Gradient descent

\[ \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \]

parabolic PDEs

\[ D(u) = \nabla \bullet (\phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|}) \]

\[ \Phi(u) = |\nabla u|^2 \quad \Rightarrow \quad D(u) = \Delta u \quad \text{Tikhonov (linear)} \]

\[ \Phi(u) = |\nabla u| \quad \Rightarrow \quad D(u) = \nabla \bullet (\frac{\nabla u}{|\nabla u|}) \quad \text{TV (non-linear)} \]
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Tikhonov and TV variational deblurring

\[ u(t = 0) = u_0 \text{ in } \Omega \]

\[ u_t = -\nabla_u E_\lambda(u; k) = -k^t \ast (k \ast u - u_0) + \lambda D(u) = 0 \text{ in } \Omega \]

Gradient descent

\[ \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \]

parabolic PDEs

\[ D(u) = \nabla \cdot (\phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|}) \]

Tikhonov: linear diffusion-reaction PDE...classical FD schemes

TV: non-linear diffusion-reaction PDE...ad-hoc num. schemes
Comparison (only blur)

Tikhonov

Total Variation
Comparison (blur and noise)

Tikhonov

Total Variation

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Blurred and Noisy Images

Total Variation Regularization

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Edge-enhanced regularization
PDE-based approaches

Linear (forward parabolic): smoothing
[Witkin-Koenderink '83-84, heat equation = Gaussian filter]

\[ u_t = \Delta u \]

\[ u_t = G \sqrt{\frac{\Delta}{2t}} * u_0 \]

Scale-space
PDE-based approaches

Non-Linear (forward-backward parabolic): smoothing-enhancing
[Perona Malik '87, Cattè Lions Morel '92, non-linear diffusion]

\[ u_t = \nabla \cdot ( g(\|\nabla u\|_2^2) \nabla u ) \]

\[ g(\|\nabla u\|^2) = \frac{1}{1 + \frac{\|\nabla u\|^2}{k^2}} \]

\[ g(\|\nabla u\|^2) = \exp\left(-\frac{\|\nabla u\|^2}{k^2}\right) \]
PDE-based approaches

Anisotropic Non-linear diffusion: smoothing-enhancing
[Weickert, 98, tensor diffusion]

\[ u_t = \nabla \bullet (D(\frac{\nabla u}{\rho})\nabla u) \]

Edge-enhancing

Coherence-enhancing
PDE-based approaches

Non-Linear (hyperbolic): enhancing
[Osher Rudin '90, shock filters]

\[ u_t = - |u_x| \text{sign}(u_{xx}) \]
Blind Deblurring MAP estimation

Maximize the posterior probability:

\[ \hat{u} = \arg \max \ p(u, k \mid u_0) \]

Minimize the Logarithmic Energy:

\[ \hat{u} = \arg \min \ E(u, k \mid u_0) = \arg \min \ ( -\log p(u, k \mid u_0)) \]

But (from Bayes rule):

\[ p(u, k \mid u_0) \propto p(u_0 \mid u, k) \ p(u, k) = p(u_0 \mid u, k) \ p(u) p(k) \]

Hence:

\[ \hat{u} = \arg \min \ ( E(u, k \mid u_0) = E(u_0 \mid u, k) + E(u) + E(k) ) \]

i.i.d Gaussian noise + exponential prior for image gradients:

\[ \hat{u}_{\lambda', \lambda} = \arg \min \left( \int_{\Omega} (k * u - u_0)^2 + \lambda \phi(|\nabla u|) + \lambda' \Psi(|\nabla k|) \right) \]
**TV Variational Blind Deblurring**

(C. and Wong (IEEE TIP, 1998))

$$\hat{u}_{\lambda, \lambda'} = \arg \min \left( \int_{\Omega} (k * u - u_0)^2 + \lambda |\nabla u| + \lambda' |\nabla k| d\Omega \right)$$

$$u, k \geq 0, \int k(x, y) dx dy = 1, \ k(x, y) = k(-x,-y)$$

- **Alternating Minimization Algorithm:**
  $$F(u^{n+1}, k^n) = \min_u F(u, k^n)$$

  $$F(u^{n+1}, k^{n+1}) = \min_k F(u^{n+1}, k)$$

- **Globally convergent with $H^1$ regularization.**
TV Variational Blind/Non-Blind Deblurring

- An out-of-focus blur is recovered automatically
- Recovered blind deconvolution images almost as good as non-blind
- Edges well-recovered in image and PSF

\[ l = 2 \times 10^{-6}, \quad l' = 1.5 \times 10^{-5} \]
TV (Blind) Deblurring, Denoising, Inpainting

Image degraded by blur, noise and missing regions

- Scratches
- Occlusions
- Defects in films/sensors
TV (Blind) Deblurring

\[ E_\lambda(u) = \int_{R^2} |\nabla u| + \lambda \int_{R^2} |k * u - u_0|^2 + \lambda' \int_{R^2} |\nabla k| \]

Out-of-focus blur

Gaussian blur (harder: zero-crossing in the frequency domain)
TV Inpainting

\[ E_\lambda(u) = \int_{\Omega=E \cup D} |\nabla u| + \lambda \int_{E} |u - u_0|^2 \]

Graffiti Removal

Scratch Removal

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TV (Blind) Deblurring and Inpainting

Problems with inpaint then deblur:

Inpaint first → reduce plausible solutions
Should pick the solution using more information
TV (Blind) Deblurring and Inpainting

Problems with deblur then inpaint:

Different BC’s correspond to different intensities in inpaint regions. Most local BC’s do not respect global geometric structures.
TV (Blind) Deblurring and Inpainting

\[ E_\lambda (u) = \int_{\Omega=E \cup D} |\nabla u| + \lambda \int_{E} |k * u - u_0|^2 + \lambda' \int_{E} |\nabla k| \]
TV (Blind) Deblurring and Inpainting

Inpaint then deblur
(many ringings)

Deblur then inpaint
(many artifacts)
TV (Blind) Deblurring and Inpainting

- the vertical strip is completed
- the vertical strip is not completed
- use higher order inpainting methods
  - E.g. Euler’s elastica, curvature driven diffusion

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(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH 2008
(Blind) Motion Deblurring

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(Blind) Motion Deblurring

Q. Shan et al., High-quality Motion Deblurring from a Single Image, SIGGRAPH2008
(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH2008

iterations
(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH 2008

Hand-held camera
(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH 2008

Hand-held camera
Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH2008

Hand-held camera
(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH2008

Hand-held camera
(Blind) Motion Deblurring

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Hand-held camera
(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH2008

Hand-held camera

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(Blind) Motion Deblurring

Q. Shan et al., *High-quality Motion Deblurring from a Single Image*, SIGGRAPH2008

Hand-held camera: non space-invariant blur (slight camera rotation, motion parallax)
(Blind) Deblurring of 1-D Bar Codes

S. Esedoglu, *Blind deconvolution of bar code signals*, Inverse Problems, 2004

$$u = \alpha \cdot G_\sigma * u_0 + n$$

- i.i.d additive Gaussian noise
- Spatially-invariant Gaussian blur
- Scalar gain
(Blind) Deblurring of 1-D Bar Codes

S. Esedoglu, *Blind deconvolution of bar code signals*, Inverse Problems, 2004

TV regularization (no a priori information on the number of bars)

\[
\min_E \ E_\lambda(u, \alpha, \sigma) = \int_R |u'| + \lambda \int_R (\alpha \cdot G_\sigma * u - u_0)^2
\]

Euler-Lagrange equations

Gradient descent
(Blind) Deblurring of 1-D Bar Codes

S. Esedoglu, *Blind deconvolution of bar code signals*, Inverse Problems, 2004
Deblurring of 2-D Bar Codes

R. Choksi et al., Anisotropic total variation regularized $L^1$-approximation and denoising/deblurring of 2D bar codes, Inverse Problems and Imaging, 2011

\[ u = k \ast u_0 + n \]

Standard isotropic TV: rounding off of corners

L2 fidelity: loss of contrast

anisotropic TV

L1 fidelity

\[
\min_{u \in R^2} E_\lambda(u; k) = \int_{R^2} |u_x| + |u_y| + \lambda \int_{R^2} |k \ast u - u_0|
\]

approximated as finite dimensional convex optimization problem

discretization by using standard forward finite differences and quadrature

Linear program reformulation
Deblurring of 2-D Bar Codes

R. Choksi et al., Anisotropic total variation regularized $L^1$-approximation and denoising/deblurring of 2D bar codes, Inverse Problems and Imaging, 2011
Blind Deblurring

Embedded Image Processing

Image Restoration: Blind deblurring
Application to barcode decoding*

* This work was supported by Realey3D, SA and Qipit, Inc
Work in progress

Variational constraint on noise whiteness

Fractional-order diffusion
Thanks for your attention!